MATH 211-2014S FINAL EXAM

NAME:

(1) circle all the statements that are correct

T F if $2+1=0$ and $1+2=2+1$ then $2+2=3$ or $2+2=4$

T F there is exactly one set A such that $|\mathcal{P}(A)| \leq 1$

T F if $A \cup C = B \cup C$ then $A = B$

T F The cardinality of $B \bigoplus B$ does not depend on the cardinality of B

T F A and B are two set such that $|A| = 3$ and $|B| = 4$. The number of one-to-one functions from A to B is $P(4,3)$

T F The bubble sort has complexity $\Theta(n^3)$

T F The coefficient of $x^{11}y^6$ in the expansion of $(x-3y)^{17}$ is 0

T F The inverse of a modulo m always exists but it may not be unique unless $\gcd(a,m) = 1$

T F The number of subsets, of a set with 10 elements, that contain more than two element is 1013

T F The characteristic equation of the recurrence relation $a_n = -a_{n-2} - a_{n-4}$ is $r^4 - r^2 - 1 = 0$

(2) There are seven girls and seven boys in a group. A committee of 6 people is to include 3 girls and 3 boys. How many committees are there such that Lama is included but Sama is not included? (where, of course, Sama and Lama are two girls in the group).

$\binom{5}{2}$ ways to get two girls besides Lama

(5 because Sama is excluded)

& $\binom{7}{3}$ ways to get 3 boys.

So we get $\binom{5}{2} \times \binom{7}{3}$

(3) A communication channel can transmit three types of signals, a, b, c. Each of the first two signals takes one microsecond to be transmitted, while the third signal takes two microseconds. So, for example, some of the messages that can be transmitted in 3 microseconds are: aaa, ac, bc, and cb. Write a recurrence relation for the number of distinct messages that can be transmitted in n microseconds if a message can only include some or all of the three signals described above. Specify any necessary initial conditions.

Initial conditions: $a_1 = 2$ (either 'a' or 'b')
$a_2 = 5$ ('aa', 'ab', 'ba', 'bb' & 'c')

For $n \geq 3$ now Any message will either start with 'a', 'b' or 'c'.
If 'a' or 'b': the rest of the message will be transmitted in $(n-1)$ microseconds: so we have $2 \times a_{n-1}$ such messages of length n. If the message starts with 'c', we use up 2 ms so we have $(n-2)$ remaining microseconds, a_{n-2} can be transmitted in these $(n-2)$ ms. So we get

$a_n = 2a_{n-1} + a_{n-2}$; $n \geq 3$.

(4) The number of bit strings that have length between 7 and 10 (inclusive) and both start and end with zeroes is

Length 7 : 2^5
——— 8 : 2^6
——— 9 : 2^7
Length 10 : 2^8

we get $2^5 + 2^6 + 2^7 + 2^8 =$

$32 + 64 + 128 + 256 = 480$

(5) The number of English strings with four letters and that have exactly two distinct letters is

Case 1 : 'xyyy' : $\binom{26}{2} \times \frac{4!}{3!\,1!} \times 2$ (order important in choosing 2 letters)

Case 2 : 'xxyy' : $\binom{26}{2} \times \frac{4!}{2!\,2!}$ (order not important)

The answer is the sum $\binom{26}{2}\left[8 + 6 \right]$

(6) Given that $a \equiv 7 \mod 12$, $b \equiv 11 \mod 12$, $c \equiv 5 \mod 12$, $d \equiv 6 \mod 12$, find the remainder of the division by 6 of $a^4 + b^4 + c^4 + d^4$

$a^2 \equiv 49 \equiv 1 \mod 12 \Rightarrow a^4 \equiv 1 \mod 12$

similarly, $b^4 \equiv 1 \mod 12$ & $c^4 \equiv 1 \mod 12$

& $\underline{d^4 \equiv 0}$ $\mod 12$ so $x = a^4 + b^4 + c^4 + d^4 \equiv 3 \mod 1$

so $x = 12K + 3 = 6(2K) + 3$

Hence $x \equiv 3 \mod 6$.

(7) A particular solution of the recurrence equation $a_n = 3a_{n-1} + 4a_{n-2} + n2^n$ is

roots of hom. ch. eq. are 4 & −1.

So Form of particular solution (since 2 is not a root) $(An+B)2^n$. We need to plug it in to get A & B.

We get $a_n^{(p)} = \left(\frac{-2}{3}n + \frac{10}{3}\right)2^n$.

(8) Show that the relation R, consisting of all pairs (x, y) where x and y are binary strings of length two or more that agree in their first two bits, is an equivalent relation on the set of all binary strings of length two or more. State explicitly the equivalence classes of this relation.

Let B_1, B_2 & B_3 be binary strings of length ≥ 2.

Reflexive: B_1 shares the first 2 bits with itself.

Symmetric: if B_1 shares the same first two bits with B_2 then B_2 shares the first two bits with B_1.

Transitive: similar.

Equivalence relations: [00], [01], [10] & [11], where [ab] represents all binary strings that start with 'ab'.

(9) Consider the relations, on the set $\{1, 2, 3, 4\}$,

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

$R_2 = \{(1,1), (1,2), (3,4)\}$

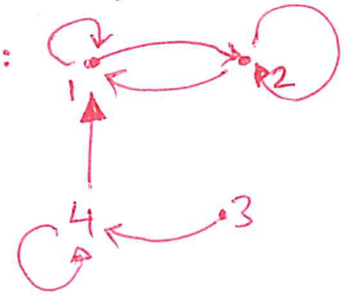
2pt (a) represent each relation both using a digraph and a matrix.

2pt (b) Write the matrices of the relations $R_1 \setminus R_2$, $R_1 \circ R_2$.

2pt. (c) Which of these relations is antisymmetric? symmetric?

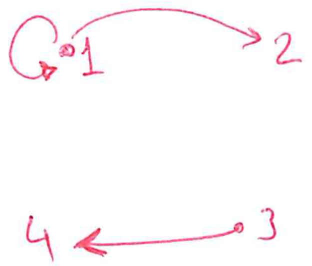
2pt. (d) Write down (your own) relation on the set $\{1, 2, 3, 4\}$ that is both symmetric and antisymmetric?

(a) R_1 :



$$M_{R_1} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

R_2



$$M_{R_2} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(b) $R_1 \setminus R_2$: $M_{R_1 \setminus R_2} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$

$$M_{R_1 \circ R_2} = M_{R_2} \odot M_{R_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) R_1 : not symmetric : $1 R_1 2$ but $2 \not R_1 1$.

not antisymmetric : $1 R_1 2$, $2 R_1 1$ but $1 \neq 2$.

R_2 : not symmetric : $1 R_2 2$ but $2 \not R_2 1$

antisymmetric.

(d) $R = \{(1,1), (2,2)\}$ OR $\{(1,1), (1,2)\}$

OR $\{(1,2), (1,3), (1,4), (2,3), (4,2), (4,3)\}$

(10) Apply the extended Euclidean Algorithm to find gcd(765, 101), then use it to calculate the inverse of 101 modulo 765

$765 = 7 \times 101 + 58$

$101 = 1 \times 58 + 43$

$58 = 1 \times 43 + 15$

$43 = 2 \times 15 + 13$

$15 = 1 \times 13 + 2$

$13 = 6 \times 2 + \underline{1}$

$2 = 2 \times 1 + 0$

$1 = 13 - 6 \times 2 = 13 - 6(15 - 13)$

$= 7 \times 13 - 6(15) = 7(43 - 2 \cdot 15) - 6 \cdot 1$

$= -20(15) + 7(43) = -20(58 - 43) + 7 \cdot 4$

$= 27(43) - 20(58) = 27(101 - 58) - 20 \times 5$

$= -47(58) + 27(101) =$

$-47(765 - 7 \times 101) + 27(101)$

$= 356(101) - 47 \times 765.$

so the inverse of 101 modulo 765 is: 356

(11) A and B are two set such that $|A| = 3$ and $|B| = 4$. Find the number of onto functions from B to A.

$3^4 - C(3,1) \cdot (3-1)^4 + C(3,2) \times (3-2)^4$

$= 81 - 3 \times 16 + 3 = 36$

(12) Suppose that the variable x represents students and y represents courses, and:
$U(y)$: y is an upper-level course $M(y)$: y is a math course $F(x)$: x is a freshman
$A(x)$: x is a part time student $T(x, y)$: student x is taking course y.
Write the statements below using these predicates and any needed quantifiers.

(a) every freshman student is taking at least two courses.
(b) there is a part time freshman student who is not taking any course.
(c) there are at least two freshmen students who are taking the same upper level math courses.

(a) $\forall x \exists y_1 \exists y_2 (F(x) \rightarrow T(x, y_1) \wedge T(x, y_2) \wedge y_1 \neq y_2)$

(b) $\exists x (F(x) \wedge A(x) \wedge \forall y \neg T(x, y))$

(c) $\exists x_1 \exists x_2 (x_1 \neq x_2 \wedge F(x_1) \wedge F(x_2) \wedge \forall y ((U(y) \wedge M(y)) \rightarrow (T(x_1, y) \leftrightarrow T(x_2, y))))$

(13) Prove that an integer n is odd unless $5n^2+7$ is odd. That is:

We need to show that if $5n^2+7$ is not odd then n is odd. By contraposition, suppose n is even then n^2 is even $\Rightarrow 5n^2$ is even & so $5n^2+7$ is odd.

(14) Find a function from $\mathbb{Z}$ to $\mathbb{N}$ that is a bijection. You will need to verify that your function is one to one and onto.

The following diagram gives the function:

$$\begin{array}{ccccccccc} 0 & -1 & +1 & -2 & +2 & -3 & +3 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{array}$$

In symbols:

$$f(z) = \begin{cases} 2z & \text{if } z \text{ is positive or zero} \\ -2z-1 & \text{if } z \text{ is negative} \end{cases}$$

f is 1-1: suppose z_1 & z_2 are such that $f(z_1)=f(z_2)$

if this value is even, then both z_1 & z_2 are positive

so $2z_1 = 2z_2 \Rightarrow z_1 = z_2$.

If this common value is odd, then z_1 & z_2 are <0

so $-2z_1-1=-2z_2-1 \Rightarrow z_1=z_2$.

f is onto: let $n \in \mathbb{N}$. If n is odd, let $z = -\frac{n+1}{2}$

then $f(z) = -2\left(-\frac{n+1}{2}\right)-1 = n+1-1 = n$.

If n is even, let $z = \frac{n}{2}$, so $f\left(\frac{n}{2}\right) = 2 \cdot \frac{n}{2} = n$.

(15) Give a recursive definition for the set of positive integers not divisible by 2 nor by 3.

Base Case: $1 \in S$, $5 \in S$

Recursive step: if $s \in S$ then $s+6 \in S$

(16) Prove by induction that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ where f_n is the n^{th} Fibonacci number.

Base Case: $n=1$: $1 = f_1^2 = f_1 \times f_2 = 1 \times 1 = 1$.

Ind. hyp. suppose, for some k, $f_1^2 + \cdots + f_k^2 = f_k f_{k+1}$

Ind. step: we need to show that the predicate is true for $(k+1)$.

$$f_1^2 + \cdots + f_k^2 + f_{k+1}^2 = f_k f_{k+1} + f_{k+1}^2$$

$$= f_{k+1}(f_k + f_{k+1}) = f_{k+1}(f_{k+2})$$

This proves the inductive step. The result now follows by the principle of mathematical induction.

(17) Given a list of integers (*not* necessarily all distinct), write a function that returns the last occurrence of the minimum. (For full credit, the function complexity must be linear, i.e. the number of computations must be $O(n)$ where n is the size of the input list).

procedure Last_occurrence ($a_1, \ldots, a_n$: integers not all distinct)

$M = a_1$
$l = 1$
for ($i = 2$ to n)
    if $a_i < M$
        $M = a_i$
        $l = i$
    else if $a_i = M$
        $l = i$
return i